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When does the ring $K[y]$ have the coefficient assignment property?

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Abstract

If K is an algebraically closed field, then it is known that $K[y]$ has the coefficient assignment property. Conversely, suppose that the field K has characteristic zero and contains the primitive nth roots of unity for all positive integers n. If $K[y]$ has the coefficient assignment property, then K is closed under taking nth roots for all positive integers n.

Let R be a commutative ring with (A,B) an n-dimensional controllable system over R. Thus, A is an $n \times n$ matrix, B is an $n \times m$ matrix, and the R-module generated by the columns of the matrix $[B, AB, \dots A^{n-1}B]$ is R^n.

If R is a field, then the controllability of a system is equivalent to any of the following three conditions:

1. There exists a matrix F and a vector v such that Bv is a cyclic vector for the matrix $A+BF$.
2. For each monic, nth degree polynomial $f(x) \in R[x]$, there exists a matrix F such that the characteristic polynomial of $A+BF = f(x)$.
3. For each collection $\{r_1, \dots, r_n\}$ of elements of R, there exists a matrix F such that the characteristic polynomial of $A+BF = (x-r_1)\cdots(x-r_n)$.

Over an arbitrary ring, these are no longer equivalent. Instead, for a system (A,B) over R, we have that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and that each of these conditions implies the controllability of the system. A ring R is called an *FC-ring* if condition (1) is satisfied for all controllable systems over R. A ring R is called a *CA-ring* if condition (2) is satisfied for all controllable systems over R. A ring R is called a *PA-ring* if condition (3) is satisfied for all controllable systems over R. Thus, any FC-ring is a CA-ring and any CA-ring is a PA-ring.

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The most general class of rings having the FC-property is the class of *local-global* rings [1]. In a very real sense, these rings behave like local rings and essentially have the FC-property because fields do. The first nontrivial example of a ring having the CA-property but not the FC-property was given in [2, 10] where it was shown that if K is an algebraically closed field and y is an indeterminate, then $K[y]$ is a CA-ring and if the characteristic of K is different from 0 , then $K[y]$ is not an FC-ring. (This work was motivated by the problem of deciding whether or not the ring $\boldsymbol{C}[y]$ is an FC-ring if $\boldsymbol{C}$ is the field of complex numbers.)

Now, it was shown in [3] that if $\boldsymbol{R}$ is the field of real numbers, then $\boldsymbol{R}[y]$ is not a CA-ring. These facts taken together suggest the following question:

For which fields K is it true that $K[y]$ is a CA-ring?

In this paper, we establish some necessary conditions on K in order for $K[y]$ to be a CA-ring. Even this small step requires considerable effort and leads us to believe that the problem is a difficult one.

Our result is the following.

Theorem 0.1. *Let K be a field with y an indeterminate over K. Let q be a prime integer different from the characteristic of K and suppose that K contains all qth roots of unity. If $K[y]$ is a CA-ring, then K is closed under taking qth roots; that is, the map $\phi : K \longrightarrow K$ defined by $\phi(x) = x^q$ is surjective. In particular, suppose that K is a field of characteristic 0 and that for each positive integer n, K contains all the nth roots of unity. If $K[y]$ is a CA-ring, then for each positive integer n, K is closed under taking nth roots.*

Proof. If $\omega \in K, \omega \neq 0$, set $\alpha = (y-1)^{q-1} \cdot (y-\omega)$ and $\beta = y + \alpha \cdot f(y)$, for $f(y)$ some polynomial in $K[y]$ to be determined later. We will apply the following surprisingly deep technical result.

Lemma 0.2. *Let k be a field with y an indeterminate over k. Let q be a prime integer different from the characteristic of k . If $\omega \in k, \omega \neq 0$, set $\alpha = (y-1)^{q-1} \cdot (y-\omega)$ and $\beta = y + \alpha \cdot f(y)$, for $f(y)$ some polynomial in $k[y]$. If α is a qth power modulo β, then ω is a qth power in k.*

Proof. First we establish some notation. Let X be a nonsingular k-variety with rational function field L. Let X_1 denote the set of points of X of codimension 1. Throughout cohomology groups and sheafs will be for the étale topology. The sheaf of units on X is denoted $\mathbb{G}_m$. The group $H^2(X, \mathbb{G}_m)$ is the cohomological Brauer group. If X is an affine scheme (for example) it is known by the Gabber–Hoobler Theorem [5] that the Brauer group $B(X)$ of classes of Azumaya $\mathcal{O}_X$-algebras is isomorphic under a canonical embedding to the torsion subgroup of $H^2(X, \mathbb{G}_m)$. The group $H^1(X, \mathbb{Z}/n)$ parametrizes the cyclic Galois extensions of X with group $\mathbb{Z}/n$.

Given units δ and γ in L^*, let n be a positive integer that is invertible in L and let ζ be a primitive nth root of unity in L. The symbol algebra $(\delta, \gamma)_n$ is the associative

L-algebra generated by elements u, v subject to the relations $u^n = \delta$, $v^n = \gamma$ and $uv = \zeta vu$. The symbol algebra $(\delta, \gamma)_n$ is central simple over L and represents a class in ${}_nB(L)$.

Given a finite-dimensional central L-division algebra D, it is possible to measure the ramification of D at any point $x \in X_1$. The local ring $\mathcal{O}_{X,x}$ at x is a discrete valuation ring. Let v be the discrete rank-1 valuation on L corresponding to the local ring $\mathcal{O}_{X,x}$. Let $k(x)$ denote the residue field at x. Assume that $k(x)$ is perfect. (If $k(x)$ is not perfect, the following still works if $(D:L)$ is prime to the characteristic of $k(x)$.) The theory of maximal orders [8, Section 5.7] associates with D a cyclic extension L of $k(x)$. Let L^v be the completion of L and D^v the division algebra component of $D \otimes L^v$. Let A be a maximal order for D^v in the complete local ring $\mathcal{O}^v_{Xx}$ and let $A(x) = A \otimes k(x)$ be the algebra of residue classes. Then $A(x)$ is a central simple algebra over L for some cyclic Galois extension $L/k(x)$. The cyclic extension $L/k(x)$ represents a class in $H^1(k(x), \mathbb{Q}/\mathbb{Z})$.

The assignment $D \mapsto L$ induces a group homomorphism

$$B(L) \to H^1(k(x), \mathbb{Q}/\mathbb{Z}) \tag{1}$$

for each discrete rank-1 valuation v on L corresponding to a point $x \in X_1$. We call L the ramification of D along x. The algebra D will ramify at only finitely many $x \in X_1$. Those x for which the cyclic extension $L/k(x)$ is nontrivial make up the so-called ramification divisor of D. So Eq. (1) induces a homomorphism

$$B(L) \xrightarrow{a} \coprod_{x \in X_1} H^1(k(x), \mathbb{Q}/\mathbb{Z}). \tag{2}$$

Let n be a positive integer. If L and $k(x)$ both contain $1/n$ and a primitive nth root of unity ζ, the homomorphism (2) agrees with the tame symbol. On the symbol algebra $(\delta, \gamma)_n$ over L, the value of the homomorphism (1) is the cyclic extension $L/k(x)$ which is obtained by adjoining the nth root of

$$(-1)^{v(\gamma)v(\delta)} \delta^{v(\gamma)} / \gamma^{v(\delta)} \tag{3}$$

to $k(x)$.

The proof of the next result of Grothendieck can be found in [6, III, Proposition 2.1] or [7, p. 107, Example 2.22, case(a)].

Proposition 0.3. *Let X be a regular integral scheme of dimension 1. Let $L = L(X)$ be the stalk at the generic point of X and X_1 the set of closed points of X. Suppose that for each $x \in X_1$, the residue field $k(x)$ is perfect. Then there is an exact sequence*

$$0 \to H^2(X, \mathbb{G}_m) \to H^2(L, \mathbb{G}_{m,L}) \xrightarrow{a} \coprod_{x \in X_1} H^1(k(x), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} H^3(X, \mathbb{G}_m) \to H^3(L, \mathbb{G}_{m,L}). \tag{4}$$

If we do not assume the residue fields are perfect, the sequence is still exact for the q-primary components of the groups, for any prime q distinct from the residue characteristics of X.

The first 2 groups in Eq. (4) are the Brauer groups of X and L, respectively. The map a in (4) is "the ramification map" (2). The fact that in (4) $r \circ a$ is the zero map can be thought of as a quadratic reciprocity law (for elements of order 2, or a qth degree reciprocity law for elements of order q). But to have practical implications, one must know that $H^1(k(x), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} H^3(X, \mathbb{G}_m)$ is injective for some $x \in X_1$. Lemma 0.4 states this is the case when X is the projective line over a field k and x is a point with residue field k — for the proof, see [4].

Lemma 0.4. *Let k be a field and n a positive integer invertible in k. Let x be a closed point of $X = \mathbb{P}^1_k$ with residue field $k(x) = k$. There exists a natural Gysin map*

$$ {}_nH^1(k(x), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} {}_n H^3(X, \mathbb{G}_m) $$

which is injective.

At last we are able to prove Lemma 0.2.

Denote by X the projective line over k, $X = \mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$. Let L be the field of rational functions on X. Dehomogenize with respect to x_1, set $y = x_0/x_1$ and view L as $k(y)$. Assume ω is not a qth power in k. We will show that α is not a qth power modulo β. The proof amounts to forcing a qth degree reciprocity law out of Proposition 0.3 for the field $k(y)$.

Consider the symbol algebra $(\alpha, \beta)_q$ as a class in ${}_qB(L)$. We show that $(\alpha, \beta)_q$ is nontrivial (is not in $\ker a$) and has nontrivial ramification. Let x be the closed point of X where $y = \omega$. At the point x, the residue field is k and the ramification of $(\alpha, \beta)_q$ corresponds to the field extension $k(1/\omega^{1/q})$, which represents an element of order q in $H^1(k(x), \mathbb{Q}/\mathbb{Z})$. By Lemma 0.4, $H^1(k(x), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} H^3(X, \mathbb{G}_m)$ is injective. However in Eq. (4), $r \circ a$ is the zero map. So there is another closed point $x' \neq x$ such that the symbol algebra $(\alpha, \beta)_q$ ramifies at x'. Notice that $(\alpha, \beta)_q$ is unramified at "the point at infinity" corresponding to $x_1 = 0$. This is because when $x_1 = 0$, α is a qth power hence the tame symbol (3) is a qth power. At the point corresponding to the other prime factor $y - 1$ of α, we see that β is equivalent to 1, hence is a qth power. So $(\alpha, \beta)_q$ is unramified at $y - 1$ also.

The symbol algebra $(\alpha, \beta)_q$ ramifies only at primes on X corresponding to irreducible factors of $\alpha\beta$ in $k[y]$ since if $h(y)$ is an irreducible polynomial in $k[y]$ which does not divide $\alpha\beta$, in the henselian local ring at the point corresponding to $h(y)$, the valuations $v(\alpha)$ and $v(\beta)$ are both equal to 0. By a process of elimination, the symbol algebra $(\alpha, \beta)_q$ necessarily ramifies at a point corresponding to a prime divisor $g(y)$ of the polynomial β. Therefore, α is not a qth power modulo $g(y)$. It follows that α is not a qth power modulo β. $\square$

We can now complete the proof of Theorem 0.1. For that, it suffices, by virtue of Lemma 0.2, to prove the following: If $K[y]$ is a CA-ring, then α is a qth power modulo β, where the $f(y)$ in β is still to be determined. Thus, let (A, B) be the q-dimensional system given by

$$A = \begin{bmatrix} 0 & 0 & . & . & . & . & 0 \\ y & 0 & . & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \\ 0 & 0 \\ . & . \\ . & . \\ . & . \\ 0 & 0 \end{bmatrix}.$$

This system is controllable since $(y, \alpha) = 1$ in $K[y]$. Let

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} & . & . & . & f_{1q} \\ f_{21} & f_{22} & f_{23} & . & . & . & f_{2q} \end{bmatrix}$$

be a generic $2 \times q$ feedback matrix over $K[y]$ and form the matrix $xI_q - (A + BF)$. Thus,

$$xI_q - (A + BF) = \begin{bmatrix} x - f_{11} & -f_{12} & -f_{13} & \cdots & -f_{1q} \\ -(y + \alpha \cdot f_{21}) & x - \alpha \cdot f_{22} & -\alpha \cdot f_{23} & \cdots & -\alpha \cdot f_{2q} \\ 0 & -1 & x & \cdots & 0 \\ . & . & . & \cdots & . \\ . & . & . & & . \\ . & . & . & \cdots & . \\ 0 & 0 & . & \cdots x & 0 \\ 0 & 0 & . & \cdots -1 & x \end{bmatrix}.$$

The characteristic polynomial of $A + BF$ is the determinant of this matrix. Since we are only interested in what happens modulo $\beta = y + \alpha \cdot f_{21}$ (where, for now we have chosen the "$f(y)$" to be f_{21}), we have to compute the determinant Δ of the matrix

$$\begin{bmatrix} x - f_{11} & -f_{12} & -f_{13} & \cdots & -f_{1q} \\ 0 & x - \alpha \cdot f_{22} & -\alpha \cdot f_{23} & \cdots & -\alpha \cdot f_{2q} \\ 0 & -1 & x & \cdots & 0 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & . & \cdots x & 0 \\ 0 & 0 & . & \cdots -1 & x \end{bmatrix}.$$

So,

$$\Delta = (x - f_{11}) \det \begin{bmatrix} x - \alpha \cdot f_{22} & -\alpha \cdot f_{23} & -\alpha \cdot f_{24} & \dots & -\alpha \cdot f_{2q} \\ -1 & x & 0 & \dots & 0 \\ 0 & -1 & x & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & . & \dots -1 & x \end{bmatrix}.$$

But, the determinant above is equal to

$$(x - \alpha \cdot f_{22}) \det \left(\begin{bmatrix} x & 0 & 0 & . & . & . & 0 \\ -1 & x & 0 & . & . & . & 0 \\ 0 & -1 & x & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & -1 & x & 0 \\ 0 & 0 & . & . & 0 & -1 & x \end{bmatrix} \right)$$

$$+\alpha \cdot f_{23} \det \left(\begin{bmatrix} -1 & 0 & 0 & . & . & . & 0 \\ 0 & x & 0 & . & . & . & 0 \\ 0 & -1 & x & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & -1 & x & 0 \\ 0 & 0 & . & . & . & -1 & x \end{bmatrix} \right)$$

$$-\alpha \cdot f_{24} \det \left(\begin{bmatrix} -1 & x & 0 & . & . & . & 0 \\ 0 & -1 & 0 & . & . & . & 0 \\ 0 & 0 & x & 0 & . & . & 0 \\ 0 & 0 & -1 & x & . & . & 0 \\ 0 & 0 & 0 & -1 & x & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & -1 & x \end{bmatrix} \right) + \cdots.$$

Consequently, it follows that Δ is given by

$$\begin{aligned} \Delta &= (x - f_{11}) \cdot [(x - \alpha \cdot f_{22}) \cdot x^{q-2} + \alpha \cdot f_{23} \cdot (-x^{q-3}) - \alpha \cdot f_{24} \cdot x^{q-4} + \cdots] \\ &= (x - f_{11}) \cdot [x^{q-1} - \alpha \cdot f_{22} \cdot x^{q-2} - \alpha \cdot f_{23} \cdot x^{q-3} - \alpha \cdot f_{24} \cdot x^{q-4} - \cdots] \\ &= x^q - (f_{11} + \alpha \cdot f_{22}) \cdot x^{q-1} + (f_{11} \cdot \alpha \cdot f_{22} - \alpha \cdot f_{23}) \cdot x^{q-2} \\ &\quad + \cdots + f_{11} \cdot \alpha \cdot f_{2q}. \end{aligned}$$

Now, if $K[y]$ is a CA-ring, there does exist a matrix $F = [f_{ij}]$ such that *char poly*$(A+BF) = x^q - \alpha$. This still holds modulo $\beta = y + \alpha \cdot f_{21}$ (where now, we really have chosen "$f(y)$" !) The calculation above then shows that modulo β we have that

$$\begin{array}{lclcccl} f_{11} & = & -\alpha\cdot f_{22}, & & & & \\ f_{11}\cdot\alpha\cdot f_{22} & = & \alpha\cdot f_{23} & \Rightarrow & \alpha\cdot f_{23} & = & -(\alpha\cdot f_{22})^2, \\ f_{11}\cdot\alpha\cdot f_{23} & = & \alpha\cdot f_{24} & \Rightarrow & \alpha\cdot f_{24} & = & (\alpha\cdot f_{22})^3, \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{11}\cdot\alpha\cdot f_{2,(q-1)} & = & \alpha\cdot f_{2,q} & \Rightarrow & \alpha\cdot f_{2,q} & = & (-1)^q(\alpha\cdot f_{22})^{q-1}, \\ f_{11}\cdot\alpha\cdot f_{2,q} & = & -\alpha & \Rightarrow & (-1)^{q+1}(\alpha\cdot f_{22})^q & = & (-1)^q\cdot\alpha \\ & & & \Rightarrow & (\alpha\cdot f_{22})^q & = & -\alpha \end{array}$$

when q is odd. Hence, for q odd, α is a qth power modulo β while for $q = 2$, $f_{11} = -\alpha\cdot f_{22}$ and $f_{11}\cdot\alpha\cdot f_{22} = -\alpha$. It follows that $(\alpha\cdot f_{22})^2 = \alpha$ and that α is a square modulo β. □

Remark 1. It is not difficult to construct an example of a field K which is closed under taking nth roots for each positive integer n, but which is not algebraically closed. Let S be the solvable closure of the field $\boldsymbol{Q}$ in the complex numbers. Since there exist extensions of $\boldsymbol{Q}$ that are not solvable, S is not algebraically closed. To see that S is closed under taking nth roots, we note that S clearly contains the abelian closure A of S and so contains all nth roots of unity. If x is an element of S, then adjoining an nth root of x to S gives an abelian extension of S and hence a solvable extension of $\boldsymbol{Q}$. It follows that x belongs to S.

Remark 2. Theorem 0.1 was known to be true under the assumption that $K[y]$ was an FC-ring (cf. [9, p. 96]).

Remark 3. A weak form of the converse of Theorem 0.1 is valid. Specifically, let K be a field with $\omega \in K$. Suppose that $x^n - \omega$ splits over K. Then $x^n - \omega$ is assignable over $K[y]$ for any reachable pair of dimension $\dot{n}$. The argument can be found in the proof of Theorem 2 of [2]. Let $x^n - \omega = (x - k_1)\cdots(x - k_n)$ for $k_1,\ldots,k_n \in K$. In the notation of that proof, in the matrix C', take : $\lambda = \lambda_1 = \cdots = \lambda_s = 0$, set $f = (x - k_1)\cdots(x - k_{r+1})$ and $\gamma = k_{r+2}$, $\beta_1 = k_{r+3},\ldots,\beta_s = k_n$. The characteristic polynomial of the appropriately transformed pair is precisely $x^n - \omega$.

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